JOURNAL OF APPROXIMATION THEORY 7, 355–364 (1973)

Polynomial Approximation and Distribution of Electrons

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Communicated by G. G. Lorentz

Received February 16, 1971

1. INTRODUCTION AND RESULTS

Let D be a bounded simply connected domain in the complex plane whose boundary is a rectifiable Jordan curve C. Let D^∞ denote the complement of the closure of D with respect to the extended plane and let Φ be the conformal map of $|z| > 1$ onto D^∞ such that $\Phi(\infty) = \infty$and $\Phi'(\infty) > 0$. Extend Φ to the topological map (also denoted by Φ) from $|z| \geqslant 1$ onto the closure of D^∞ and let $\phi(t) = \Phi(e^{it})$. By the rectifiability of C, it is known (and follows easily from the F. and M. Riesz theorem) that ϕ is an absolutely continuous function. For convenience, we assume that D contains the origin.

DISTRIBUTION OF ELECTRONS

Let $z_{n,k} = \phi(\theta + 2\pi k/n), k = 1,..., n; n = 1, 2,...$, where θ is an arbitrarly chosen real number, and let

$$p_n(z) = \prod_{k=1}^{n} (1 - z/z_{n,k}).$$

The points $z_{n,k}$, $k = 1,..., n$ and $n = 1, 2,...$ are called the Fejèr points of C and the polynomials p_n are the corresponding Fejèr polynomials normalized to be one at the origin. Since ϕ is absolutely continuous, the Fejèr polynomials p_n converge uniformly on each compact subset of D to the constant 1 [4, 6]. Or equivalently, the Fejèr points $\{z_{n,k}\}$ of C are asymptotically neutrally distributed relative to D [3], i.e.,

$$\sum_{k=1}^{n} 1/(z - z_{n,k}) \to 0$$

uniformly on every compact subset of D as $n \to \infty$.

Indeed, if $w_{n,k} \in C$, $k = 1,..., n$ and $n = 1, 2,...$, then the (negative complex conjugate of the) sum

$$s_n(z) = \sum_{k=1}^{n} 1/(z - w_{n,k})$$

represents the electrostatic field at the point z due to the electrons of unit charges at the points $w_{n,k}$, $k = 1,..., n$. Hence, $\{w_{n,k}\}$ are asymptotically neutrally distributed relative to D if and only if the "fields" s_n are asymptotically zero on each closed subset of D.

DEFINITION. Let $w_{n,k} \in C$, $k = 1,..., n$ and $n = 1, 2,...$. We say that $\{w_{n,k}\}$ are *asymptotically neutrally and boundedly distributed relative to D* (ANBD) if $\{w_{n,k}\}$ are asymptotically neutrally distributed relative to D and there exists an $M < \infty$ such that

$$\max_{z \in C} \left| \prod_{k=1}^{n} (1 - z/w_{n,k}) \right| < M$$

for all n. If such sequences $\{w_{n,k}\}$ exist on C, we say that the curve C is of class ANBD.

We observe that there exist asymptotically neutrally distributed sequences which are not ANBD. For example, let $\{z_{n,k}\}$ be Fejèr points of C, $m = n^2$ and $w_{m,1} = \cdots = w_{m,n} = z_{n,1}, ..., w_{m,m-n+1} = \cdots = w_{m,n} = z_{n,n}$. If C is so smooth that ϕ is twice continuously differentiable, then it can be shown [5] that uniformly on each compact subset of D,

$$\sum_{k=1}^{n} 1/(z - z_{n,k}) = o(1/n);$$

and hence, $\{w_{m,k}\}$ are also asymptotically neutrally distributed relative to D, although n electrons of unit charges are concentrated at each $z_{n,k}$, $k = 1,..., n$. However, if the p_n denote the Fejèr polynomials as defined previously, then

$$q_m(z) = \prod_{k=1}^{m} (1 - z/w_{m,k}) = p_n{}^n(z).$$

Since $p_n \to 1$ uniformly on compact subsets of D and all the zeros of p_n lie on C, we can see that $\lim\inf \max_C | p_n | > 1$, so that $\max_C | q_m | \to \infty$.

Hence, we have the following problem: *What curves C are of class* ANBD, *and if C is of class* ANBD, *what sequences of points on C are* ANBD?

DEFINITION. Let L be the length of the rectifiable Jordan curve C and let

$z = h(s)$, $0 \leqslant s \leqslant L$, where s denotes arc length, be a parametric representation of C. Let $0 < \alpha < 1$. Then the curve C is said to be of class $H(1, \alpha)$, if C has a continuously turning tangent line, and h' satisfies a Hölder condition of order α:

$$| h'(s) - h'(t)| \leqslant K | s - t |^{\alpha}$$

or all $s, t \in [0, L]$, where $K < \infty$.

We will establish the following theorem.

Main Theorem. *Let the* Jordan *curve C be of class $H(1, \alpha)$ where $0 < \alpha < 1$. For each $n = 1, 2, \ldots$, let $t_k = t_{n,k}$, $k = 1, \ldots, n + 1$, be points such that $0 \leqslant t_1 < \cdots < t_n < 2\pi$, $t_{n+1} = 2\pi + t_1$ and*

$$\max_{1 \leqslant j \leqslant n} (t_{j+1} - t_j) / \min_{1 \leqslant j \leqslant n} (t_{j+1} - t_j) \leqslant A,$$

where $A < \infty$. Let *$\alpha_j = \alpha_{n,j} = n(t_{j+1} - t_j)/2\pi$ and $\theta_j = \theta_{n,j} = (t_{j+1} + t_j)/2$, for $j = 1, \ldots, n$, and define*

$$q_n(z) = \prod_{j=1}^{n} (1 - z/\phi(\theta_j))^{\alpha_j}.$$

Then there is a positive constant B, independent of the choice of the $\{t_{n,j}\}$, such that

$$\max_{z \in C} | q_n(z)| \leqslant B$$

for all n.

As a trivial consequence of this theorem, we have the following

Corollary. *If C is of class $H(1, \alpha)$, $0 < \alpha < 1$, then the Fejèr points of C are* ANBD.

Hence, all Jordan curves of class $H(1, \alpha)$, $0 < \alpha < 1$, are of class ANBD.

2. Proof of the Main Theorem

We need the following four lemmas.

Lemma 1. *For each $z \in D$, we have*

$$\int_0^{2\pi} \log(1 - z/\phi(t))\, dt = 0, \tag{1}$$

where the branch of the logarithm is taken so that $\log 1 = 0$.

The proof of this is clear if we note that $\Phi(\infty) = \infty$. The following result is due to Kellogg and can be found in [7].

LEMMA 2. *Let C be of class $H(1, \alpha)$, $0 < \alpha < 1$. Then the derivative Φ' is zero free for $|z| \geqslant 1$ and Φ' satisfies a Hölder condition of order α on the unit circle*:

$$|\Phi'(e^{is}) - \Phi'(e^{it})| \leqslant K^* |s - t|^{\alpha},$$

where $K^* < \infty$ and $s, t \in [0, 2\pi]$.

As a consequence of this, we have the following

LEMMA 3. *Let C be of class $H(1, \alpha)$, $0 < \alpha < 1$. Then there exist positive constants C_1, C_2, C_3 such that for all $s, t \in [0, 2\pi]$,*

$$|\phi(s) - \phi(t)| \geqslant C_1 |s - t|, \qquad \textit{where} \quad |s - t| \leqslant \pi, \tag{2}$$

$$|\phi(s) - \phi(t)| \leqslant C_2 |s - t|, \qquad \textit{and} \tag{3}$$

$$|\phi(s) - \phi(t) - \phi'(t)(s - t)| \leqslant C_3 |s - t|^{1+\alpha}. \tag{4}$$

Proof. By the continuity of ϕ' and Kellogg's result, we have

$$\min_{0 \leqslant t \leqslant 2\pi} |\phi'(t)| > 0,$$

and hence, (2) follows. Now,

$$\begin{aligned} |\phi'(s) - \phi'(t)| &= |e^{is}\Phi'(e^{is}) - e^{it}\Phi'(e^{it})| \\ &\leqslant |e^{is}\Phi'(e^{is}) - e^{is}\Phi'(e^{it})| + |e^{is}\Phi'(e^{it}) - e^{it}\Phi'(e^{it})| \\ &\leqslant K^* |s - t|^{\alpha} + |\Phi'(e^{it})| \, |s - t| \\ &\leqslant C_3 |s - t|^{\alpha}; \end{aligned}$$

and suppose that $s < t$, then

$$\phi(t) - \phi(s) - \phi'(s)(t - s) = \int_s^t (\phi'(\tau) - \phi'(s)) \, d\tau.$$

Hence, (4) follows and (3) is a trivial consequence of (4).

LEMMA 4. *Let C be such that ϕ satisfies (2) and (3). Then there is a positive constant R such that for each β, $0 < \beta \leqslant \pi/4$ and all z in the closure of D, we have*

$$\log |1 - z/\phi(0)| - \frac{1}{2\beta} \int_{-\beta}^{\beta} \log |1 - z/\phi(t)| \, dt \leqslant R. \tag{5}$$

Proof. By the maximum principle it is sufficient to consider $z = \phi(\theta)$, and by symmetry, we let $0 < \theta \leqslant \pi$. When $0 < \beta \leqslant \pi/4$ and $\pi/2 \leqslant \theta \leqslant \pi$, (5) is trivial. Hence, we assume that $0 < \theta \leqslant \pi/2$. Now,

$$\log | 1 - \phi(\theta)/\phi(0)| - \frac{1}{2\beta} \int_{-\beta}^{\beta} \log | 1 - \phi(\theta)/\phi(t)| \, dt$$
$$= \frac{1}{2\beta} \int_{-\beta}^{\beta} \log \left| \frac{\phi(0) - \phi(\theta)}{\phi(t) - \phi(\theta)} \right| dt + \frac{1}{2\beta} \int_{-\beta}^{\beta} \log \left| \frac{\phi(t)}{\phi(0)} \right| dt.$$

The integral

$$\frac{1}{2\beta} \int_{-\beta}^{\beta} \log | \phi(t)/\phi(0)| \, dt$$

is clearly bounded above. Also, since $0 < \beta \leqslant \pi/4$ and $0 < \beta \leqslant \pi/2$, $| \theta - t | < \pi$ for $-\beta \leqslant t \leqslant \beta$, and by (2) and (3), we have

$$\left| \frac{\phi(0) - \phi(\theta)}{\phi(t) - \phi(\theta)} \right| \leqslant \frac{C_2}{C_1} \left| \frac{\theta}{t - \theta} \right|.$$

Hence, we obtain

$$\log \left| 1 - \frac{\phi(\theta)}{\phi(0)} \right| - \frac{1}{2\beta} \int_{-\beta}^{\beta} \log \left| 1 - \frac{\phi(\theta)}{\phi(t)} \right| dt$$
$$\leqslant R_1 + \log \frac{C_2}{C_1} + \frac{1}{2\beta} \int_{-\beta}^{\beta} \log | \theta/(t - \theta)| \, dt.$$

By a proof similar to that of Lemma 4.1 in [2], we can show that

$$\frac{1}{2\beta} \int_{-\beta}^{\beta} \log | \theta/(t - \theta)| \, dt \leqslant 1.$$

For all $0 < \beta \leqslant \pi/4$ and $0 < \theta \leqslant \pi$. The proof of the lemma is completed by letting $R = R_1 + \log(C_2/C_1) + 1$.

With the above lemmas, we can prove the main theorem. Let $\Delta_j = (t_{j+1} - t_j)/2, j = 1,..., n$. By Lemma 1, we obtain, using the principal values of the logarithms,

$$\log q_n(z) = \sum_{j=1}^{n} \alpha_j \log \left(1 - \frac{z}{\phi(\theta_j)}\right) - \frac{n}{2\pi} \int_0^{2\pi} \log \left(1 - \frac{z}{\phi(t)}\right) dt$$
$$= -\sum_{j=1}^{n} \frac{n}{2\pi} \int_{-\Delta_j}^{\Delta_j} \log \left\{1 - \frac{z/\phi(\theta_j + t) - z/\phi(\theta_j)}{1 - z/\phi(\theta_j)}\right\} dt. \qquad (6)$$

By the maximum principle, we can assume that $z = \phi(\theta)$, and without loss of generality, we restrict ourselves to the case where $t_n - 2\pi \leqslant \theta \leqslant t_1$. Now, let

$$S_1 = \{j: j \geqslant 2 \text{ and } 0 \leqslant \theta_j - \theta \leqslant \pi\}$$

and

$$S_2 = \{j: j \leqslant n - 1 \text{ and } \pi \leqslant \theta_j - \theta \leqslant 2\pi\}.$$

Then for $j \in S_1$, (2) of Lemma 3 implies that

$$|\phi(\theta_j) - \phi(\theta)| \geqslant C_1 |\theta_j - \theta| \geqslant C_1(t_j - t_1) = C_1 \sum_{k=1}^{j-1} (t_{k+1} - t_k).$$

Note that

$$\min_{1 \leqslant k \leqslant n} (t_{k+1} - t_k) \leqslant 2\pi/n \leqslant \max_{1 \leqslant k \leqslant n} (t_{k+1} - t_k),$$

so that by the hypothesis, we get

$$\max_{1 \leqslant k \leqslant n} (t_{k+1} - t_k) \leqslant 2\pi A/n$$

and

$$\min_{1 \leqslant k \leqslant n} (t_{k+1} - t_k) \geqslant 2\pi/nA.$$

Hence, for $j \in S_1$, we have

$$|\phi(\theta_j) - \phi(\theta)| \geqslant 2\pi C_1(j - 1)/nA.$$

Similarly, we can prove that for $j \in S_2$,

$$\begin{aligned} |\phi(\theta_j) - \phi(\theta)| &\geqslant |\phi(2\pi + \theta) - \phi(\theta_j)| \\ &\geqslant C_1(2\pi + \theta - \theta_j) \geqslant C_1(t_n - t_{j+1}) \\ &= C_1 \sum_{k=j+1}^{n-1} (t_{k+1} - t_k) \\ &\geqslant 2\pi C_1(n - j - 1)/nA. \end{aligned}$$

On the other hand, for $-\Delta_j \leqslant t \leqslant \Delta_j$, (3) of lemma 3 implies that

$$|\phi(\theta_j + t) - \phi(\theta_j)| \leqslant 2\pi C_2 A/2n. \tag{7}$$

Let p be the positive integer $p = [C_2 d_1 A^2/C_1 d_2] + 2$, where d_1 denotes the diameter of D and d_2 denotes the distance from 0 to C. Combining the above estimates, we see that for $p \leqslant j \leqslant n - p$,

$$|\phi(\theta_j) - \phi(\theta)| \geqslant 2\pi C_1(p - 1)/nA \geqslant (d_1/d_2) \cdot 2\pi C_2 A/n.$$

But $| \phi(\theta)/\phi(\theta_j + t)| \leqslant d_1/d_2$. Hence, for $-\Delta_j \leqslant t \leqslant \Delta_j$, $p \leqslant j \leqslant n - p$, we obtain, by using (7),

$$\begin{aligned} \left| \frac{\phi(\theta)}{\phi(\theta_j + t)} - \frac{\phi(\theta)}{\phi(\theta_j)} \right| &\leqslant \frac{d_1}{d_2} \left| 1 - \frac{\phi(\theta_j + t)}{\phi(\theta_j)} \right| \\ &\leqslant \frac{n \mid \phi(\theta_j) - \phi(\theta)|}{2\pi C_2 A} \left| 1 - \frac{\phi(\theta_j + t)}{\phi(\theta_j)} \right| \\ &\leqslant \frac{1}{2} \left| 1 - \frac{\phi(\theta)}{\phi(\theta_j)} \right|. \end{aligned} \tag{8}$$

We now split the sum in (6) into two parts:

$$\log q_n(e^{i\theta}) = \Sigma' + \Sigma'',$$

where Σ' denotes the sum over $1 \leqslant j \leqslant p - 1$ and $n - p + 1 \leqslant j \leqslant n$ and Σ'' is the sum over $p \leqslant j \leqslant n - p$. Assuming that n is so large that $\Delta_j \leqslant \pi/4$ for all j, we can use Lemma 4 to get

$$\begin{aligned} \operatorname{Re} \Sigma' &= \frac{n}{\pi} \Sigma' \Delta_j \left\{ \log \mid 1 - \phi(\theta)/\phi(\theta_j)| - \frac{1}{2\Delta_j} \int_{-\Delta_j}^{\Delta_j} \log \left| 1 - \frac{\phi(\theta)}{\phi(\theta_j + t)} \right| dt \right\} \\ &= \frac{n}{\pi} \Sigma' \Delta_j R \leqslant 2pRA. \end{aligned}$$

To study Σ'', we set

$$\chi = \frac{\phi(\theta)/\phi(\theta_j + t) - \phi(\theta)/\phi(\theta_j)}{1 - \phi(\theta)/\phi(\theta_j)},$$

so that by (8) for $p \leqslant j \leqslant n - p$ and $-\Delta_j \leqslant t \leqslant \Delta_j$, $| \chi | \leqslant 1/2$. For the same range of t and j,

$$-\log(1 - \chi) = \chi + \{(\chi^2/2) + (\chi^3/3) + \cdots\} = \chi + (\chi^2/2)\{1 + (2\chi/3) + \cdots\},$$

so that

$$-\log | 1 - \chi | \leqslant \operatorname{Re} \chi + \tfrac{1}{2} | \chi |^2/(1 - | \chi |) \leqslant \operatorname{Re} \chi + | \chi |^2.$$

Hence, we have

$$\begin{aligned} \operatorname{Re} \Sigma'' &\leqslant \frac{n}{2\pi} \sum_{j=p}^{n-p} \int_{-\Delta_j}^{\Delta_j} \operatorname{Re} \chi \, dt + \frac{n}{2\pi} \sum_{j=p}^{n-p} \int_{-\Delta_j}^{\Delta_j} | \chi |^2 \, dt \\ &= Q_1 + Q_2, \end{aligned}$$

say, where

$$Q_1 = \frac{n}{2\pi} \sum_{j=p}^{n-p} \int_{-\Delta_j}^{\Delta_j} \mathrm{Re} \left\{ \frac{\phi(\theta)/\phi(\theta_j + t) - \phi(\theta)/\phi(\theta_j)}{1 - \phi(\theta)/\phi(\theta_j)} \right\} dt$$

$$= \frac{n}{2\pi} \sum_{j=p}^{n-p} \int_{-\Delta_j}^{\Delta_j} \mathrm{Re} \left\{ \frac{\phi(\theta_j) - \phi(\theta_j + t)}{\phi(\theta_j) - \phi(\theta)} \cdot \frac{\phi(\theta)}{\phi(\theta_j)} \right\} dt$$

$$+ \frac{n}{2\pi} \sum_{j=p}^{n-p} \int_{-\Delta_j}^{\Delta_j} \mathrm{Re} \left\{ \frac{\phi(\theta_j) - \phi(\theta_j + t)}{\phi(\theta_j) - \phi(\theta)} \left(\frac{\phi(\theta)}{\phi(\theta_j + t)} - \frac{\phi(\theta)}{\phi(\delta_j)} \right) \right\} dt$$

$$= Q_1' + Q_1'',$$

where

$$Q_1' = \frac{n}{2\pi} \sum_{j=p}^{n-p} \int_{-\Delta_j}^{\Delta_j} \mathrm{Re} \left\{ \frac{-\phi'(\theta_j)}{\phi(\theta_j) - \phi(\theta)} \cdot \frac{\phi(\theta)}{\phi(\theta_j)} \right\} t\, dt$$

$$+ \frac{n}{2\pi} \sum_{j=p}^{n-p} \int_{-\Delta_j}^{\Delta_j} \mathrm{Re} \left\{ \frac{\phi(\theta_j) - \phi(\theta_j + t) + \phi'(\theta_j)t}{\phi(\theta_j) - \phi(\theta)} \cdot \frac{\phi(\theta)}{\phi(\theta_j)} \right\} dt.$$

The first sum is clearly zero since t is an odd function. By (4) of Lemma 3, we obtain

$$Q_1' \leqslant \frac{n}{2\pi} \sum_{j=p}^{n-p} \int_{-\Delta_j}^{\Delta_j} \frac{C_3 \,|\, t \,|^{1+\alpha}}{|\, \phi(\theta_j) - \phi(\theta)|} \left| \frac{\phi(\theta)}{\phi(\theta_j)} \right| dt.$$

Let $S_1' = \{j\colon p \leqslant j \leqslant n - p \text{ and } 0 \leqslant \theta_j - \theta \leqslant \pi\}$ and

$$S_2' = \{j\colon p \leqslant j \leqslant n - p \text{ and } \pi \leqslant \theta_j - \theta \leqslant 2\pi\}.$$

Then we have

$$Q_1' \leqslant \frac{n}{\pi} \frac{C_3 d_1}{d_2} \sum_{j \in S_1'} \int_0^{\Delta_j} \frac{t^{1+\alpha}}{|\, \phi(\theta_j) - \phi(\theta)|} dt$$

$$+ \frac{n}{\pi} \frac{C_3 d_1}{d_2} \sum_{j \in S_2'} \int_0^{\Delta_j} \frac{t^{1+\alpha}}{|\, \phi(\theta_j) - \phi(\theta)|} dt.$$

But $S_1' \subset S_1$ and $S_2' \subset S_2$, so that we have

$$Q_1' \leqslant \frac{n}{\pi} \frac{C_3 d_1}{d_2} \frac{1}{2 + \alpha} \frac{A}{C_1} \frac{n}{2\pi} \left(\frac{A\pi}{n} \right)^{2+\alpha} \left\{ \sum_{j \in S_1'} \frac{1}{j - 1} + \sum_{j \in S_2'} \frac{1}{n - j - 1} \right\}.$$

The right side tends to zero as fast as $\log n/n^{\alpha}$. Also, by similar reasonings, we have

$$Q_1'' \leqslant \frac{n}{2\pi} \sum_{j=p}^{n-p} \int_{-\Delta_j}^{\Delta_j} \frac{C_2^2 t^2}{|\phi(\theta_j) - \phi(\theta)|} \cdot \frac{d_1}{d_2} dt$$

$$= \frac{C_2^2 d_1}{d_2^2} \frac{2}{3} \frac{n}{2\pi} \sum_{j=p}^{n-p} \frac{\Delta_j^3}{|\phi(\theta_j) - \phi(\theta)|}.$$

Hence, again by similar reasonings as above, the upper bound of Q_1'' is of order $O(\log n/n)$. Therefore, the upper bounds for $Q_1 = Q_1' + Q_1''$ can be made as small as we please by taking n sufficiently large. Also, by applying Lemma 3 again, we have

$$Q_2 = \frac{n}{2\pi} \sum_{j=p}^{n-p} \int_{-\Delta_j}^{\Delta_j} \left| \frac{\phi(\theta)/\phi(\theta_j + t) - \phi(\theta)/\phi(\theta_j)}{1 - \phi(\theta)/\phi(\theta_j)} \right|^2 dt$$

$$\leqslant \frac{n}{2\pi} \frac{d_1^2}{d_2^2} \sum_{j=p}^{n-p} \int_{-\Delta_j}^{\Delta_j} \left| \frac{\phi(\theta_j + t) - \phi(\theta_j)}{\phi(\theta_j) - \phi(\theta)} \right|^2 dt$$

$$\leqslant \frac{n}{2\pi} \left(\frac{d_1 C_2 A n}{d_2 C_1 2\pi} \right)^2 \left(\frac{A\pi}{n} \right)^3 \left\{ \sum_{j \in S_1'} \frac{1}{(j-1)^2} + \sum_{j \in S_2'} \frac{1}{(n-j-1)^2} \right\}$$

$$< (A^5/2)(\pi d_1 C_2 / 6 d_2 C_1)^2.$$

Thus, we may take

$$\log B = 2RA((d_1 C_2 / d_2 C_1) A^2 + 2) + A^5/2(\pi d_1 C_2 / 6 d_2 C_1)^2 + 1$$

to complete the proof of the theorem.

Remark. Professor Kövari pointed out to the author that he and Pommerenke proved independently that if D is convex and $z_{n,k}$, $k = 1,\ldots, n$, are Fejèr points on C, then

$$\max_{z \in C} \prod_{k=1}^{n} |1 - z/z_{n,k}| \leqslant 4.$$

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